Modulation and Hilbert Space Representations for Rieffel's Pseudodifferential Calculus

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Abstract

We define modulation maps and modulation spaces of symbols suited to the study of Rieffel's pseudodifferential calculus. They are used to generate Hilbert space representations for the quantized C^*-algebras starting from covariant representations of the corresponding twisted C^*-dynamical system.

Introduction

In order to provide a unified framework for a large class of examples in deformation quantization, Marc Rieffel [25] significantly extended the basic part of the Weyl pseudodifferential calculus. Rieffel's calculus starts from the action Θ of a finite-dimensional vector space Ξ on a C^*-algebra $\mathcal{A}$, together with a skew-symmetric linear operator $J : \Xi \to \Xi$ that serves to twist the product on $\mathcal{A}$. Using J one defines first a new composition law $\#$ on the set of smooth elements of $\mathcal{A}$ under the action and then a completion is taken in a suitable C^*-norm. The outcome is a new C^*-algebra $\mathfrak{A}$, also endowed with an action of the vector space Ξ. Very fortunately, the corresponding subspaces of smooth vectors under the two actions, $\mathcal{A}^\infty$ and $\mathfrak{A}^\infty$, respectively, coincide. In [25] the functorial properties of the correspondence $\mathcal{A} \mapsto \mathfrak{A}$ are studied in detail and many examples are given. It is also shown that one gets a strict deformation quantization [26] of a natural Poisson structure defined on $\mathcal{A}^\infty$ by the couple (Θ, J).

If the initial C^*-algebra $\mathcal{A}$ is Abelian with Gelfand spectrum Σ, one can view Rieffel's formalism as a generalized version of the Weyl calculus associated to the topological dynamical system (Σ, Θ, Ξ) and the elements of $\mathfrak{A}$ as generalized pseudodifferential symbols. The standard form is recovered essentially when $\Sigma = \Xi$ and Θ is the action of the vector group Ξ on itself by translations. So, aside applications in Deformation Quantization and Noncommutative Geometry, one might want to use Rieffel's calculus for purposes closer to the traditional theory of pseudodifferential operators. In [19], relying on the strong functorial connections between "the classical data" (Σ, Θ, Ξ) and the quantized algebra $\mathfrak{A}$, we used the formalism to solve several problems in spectral theory. Many other potential applications are in view; their success depends partly of our ability to supply families of function spaces suited to the calculus. Since Hörmader-type symbol spaces seem to be rather difficult to define and use, we turned our attention to the problem of adapting modulation spaces to this general context.

Modulation spaces are Banach function spaces introduced long ago by H. Feichtinger [8, 9] and already useful in many fields of pure and applied mathematics. They are defined by imposing suitable norm-estimates on a certain type of transformations of the function one studies. These transformations involve a combination of translations and multiplications with phase factors.

After J. Sjöstrand rediscovered one of these spaces in the framework of pseudodifferential operators [27], the interconnection between modulation spaces and pseudodifferential theory developed considerably, cf [7, 11, 12, 13, 14, 16, 18, 28, 29] and references therein. The modulation strategy supplies both valuable symbol spaces used for defining the pseudodifferential operators and good function spaces on which these operators apply. From several points of view, the emerging theory is simpler and sharper than that relying

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on "traditional function spaces". Extensions to pseudodifferential operators constructed on locally compact Abelian groups are available [15]. In [21] the case of the magnetic Weyl calculus [20, 17] is considered in a modulation setting, while in [1, 2, 3, 4] (building on previous work of N. Pedersen [22]), modulation spaces are defined and studied for the magnetic Weyl calculus defined by representations of nilpotent Lie groups.

In this article we start the project of defining and using modulation spaces adapted to Rieffel's quantization. Only general constructions will be presented here, making efforts not to exclude the case of a non-commutative "classical" algebra $\mathcal{A}$. Extensions, more examples and a detailed study of the emerging spaces will be presented elsewhere.

For simplicity, and because this is the most interesting case, we are going to assume that the linear skew-symmetric operator J is non-degenerate, so it defines on Ξ a symplectic form $[\![\cdot,\cdot]\!]$. Thus the starting point is the quadruplet $(\mathcal{A},\Theta,\Xi,\kappa)$, where $(\mathcal{A},\Theta,\Xi)$ is the initial C^*-dynamical system and κ is the group 2-cocycle canonically given by the symplectic form (cf. (1.1)). In the first Section we recall briefly the two canonical C^*-algebras associated to such a data: Rieffel's algebra $\mathfrak{A}$ [25, 26] with composition $\#$ and the twisted crossed product algebra $\mathcal{A}\rtimes^{\kappa}_{\Theta}\Xi$ [23, 24] with composition $\diamond$. To compare them, one "doubles" the initial data, relying on a second simple dynamical system in which Ξ acts on itself by translations.

Section 2 contains the basic constructions. Very roughly, the modulation strategy starts by defining linear injective maps (called *modulation maps*) from the smooth algebra $\mathfrak{A}^\infty$ to the twisted crossed product $\mathcal{A}\rtimes^{\kappa}_{\Theta}\Xi$, indexed by "windows" belonging to Schwartz space $\mathcal{S}(\Xi)$. We insist that, for self-adjoint idempotent windows, these maps should be morphisms of *-algebras, at the price of deviating to a certain extent from the previous definitions, given for the usual Weyl calculus. Actually, all these morphisms are shadows of a single isomorphism sending the *-algebra $\mathcal{S}(\Xi;\mathfrak{A}^\infty)$ (with the Rieffel-type structure for the doubled classical data) to $\mathcal{S}(\Xi;\mathcal{A}^\infty)$ seen as a *-subalgebra of $\mathcal{A}\rtimes^{\kappa}_{\Theta}\Xi$. In addition, they extend to embeddings of $\mathfrak{A}$ in the twisted crossed product $\mathcal{A}\rtimes^{\kappa}_{\Theta}\Xi$. We use these modulation maps to induce norms on $\mathfrak{A}^\infty$ from norms defined on $\mathcal{S}(\Xi;\mathcal{A}^\infty)$. As a reward for our care to preserve algebraic structure, one gets in this way Banach algebra norms from Banach algebra norms, C^*-norms from C^*-norms, etc. In particular, Rieffel's algebra $\mathfrak{A}$ is presented as the modulation space induced from $\mathcal{A}\rtimes^{\kappa}_{\Theta}\Xi$. We also address the problem of independence of the resulting Banach spaces under the choice of the window.

In Section 3 we treat morphisms and representations. To an equivariant morphism between two classical data $(\mathcal{A}_1,\Theta_1)$ and $(\mathcal{A}_2,\Theta_2)$ one assigns canonically a morphism acting between the quantized C^*-algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ as well as a morphism acting between the two twisted crossed products $\mathcal{A}_1\rtimes^{\kappa}_{\Theta_1}\Xi$ and $\mathcal{A}_2\rtimes^{\kappa}_{\Theta_2}\Xi$. The modulation mappings intertwine these two morphisms. This has an obvious consequence upon the connection between short exact sequences of Rieffel quantized algebras and the corresponding short exact sequences of twisted crossed products. Then we turn to Hilbert space representations, surely useful if applications are aimed in the future. By using localization with respect to idempotent windows, the *-representations of the twisted crossed product $\mathcal{A}\rtimes^{\kappa}_{\Theta}\Xi$ (indexed by covariant representations of the twisted C^*-dynamical system $(\mathcal{A},\Theta,\Xi,\kappa)$) automatically supply *-representations of the smooth algebra $(\mathfrak{A}^\infty,\#)$ which extend to full C^*-representations of $(\mathfrak{A},\#)$. This also allows one to express the norm in $\mathfrak{A}$ (initially defined by Hilbert module techniques) in a purely Hilbert space language.

In the Abelian case we have previously listed families of Schrödinger-type representations in the Hilbert space $L^2(\mathbb{R}^n)$ defined by orbits of the topological dynamical system (Σ,Θ,Ξ). They were used in [19] in the spectral analysis of Quantum Hamiltonians. We are going to show in Section 4 that their Bargmann transforms can be obtained from some canonical representations of the twisted crossed product applied to symbols defined by the modulation maps. We also prove orthogonality relations, relying on a choice of an invariant measure on Σ.

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1 Involutive algebras associated to a twisted dynamical system

We shall recall briefly, in a slightly particular setting, some constructions and results concerning twisted crossed products algebras and Rieffel's pseudodifferential calculus.

The common starting point is a $2n$-dimensional real vector space Ξ endowed with a symplectic form $[\![\cdot,\cdot]\!]$. When needed we shall suppose that $\Xi = \mathscr{X} \times \mathscr{X}^*$, with $\mathscr{X}^*$ the dual of the n-dimensional vector space $\mathscr{X}$ and that for $X := (x,\xi)$, $Y := (y,\eta) \in \Xi$, the symplectic form reads $[\![X,Y]\!] := y\cdot\xi - x\cdot\eta$.

An action Θ of Ξ by automorphisms of a (maybe non-commutative) C^*-algebra $\mathcal{A}$ is also given. For $(f,X) \in \mathcal{A}\times\Xi$ we are going to use the notations $\Theta(f,X) = \Theta_X(f) = \Theta_f(X) \in \mathcal{A}$ for the X-transformed of the element f. This action is assumed strongly continuous, i.e. for any $f \in \mathcal{A}$ the mapping $\Xi \ni X \mapsto \Theta_X(f) \in \mathcal{A}$ is continuous. The initial object, containing *the classical data*, is a quadruplet $(\mathcal{A},\Theta,\Xi,[\![\cdot,\cdot,]\!])$ with the properties defined above.

To arrive at twisted crossed products, we define

$$\kappa : \Xi\times\Xi \to \mathbb{T} := \{\lambda\in\mathbb{C} \mid |\lambda| = 1\}\,, \qquad \kappa(X,Y) := \exp\left(-\frac{i}{2}[\![X,Y]\!]\right) \tag{1.1}$$

and notice that it is a group 2-cocycle, i.e. for all $X,Y,Z \in \Xi$ one has

$$\kappa(X,Y)\,\kappa(X+Y,Z) = \kappa(Y,Z)\,\kappa(X,Y+Z)\,, \qquad \kappa(X,0) = 1 = \kappa(0,X)\,.$$

Thus the classical data converts into $(\mathcal{A},\Theta,\Xi,\kappa)$, a very particular case of *twisted C^*-dynamical system* [23, 24].

To any twisted C^*-dynamical system one associates canonically a C^*-algebra $\mathcal{A}\rtimes^{\kappa}_{\Theta}\Xi$ (called *twisted crossed product*). This is the enveloping C^*-algebra of the Banach *-algebra $\big(L^1(\Xi;\mathcal{A}),\diamond,^{\diamond},\|\cdot\|_1\big)$, where

$$\|\, G\,\|_1 := \int_\Xi dX\,\|\,G(X)\,\|_{\mathcal{A}}, \qquad G^{\diamond}(X) := G(-X)^*$$

and (symmetrized version of the standard form, cf. Remark 3.3)

$$(G_1\diamond G_2)(X) := \int_\Xi dY\,\kappa(X,Y)\,\Theta_{(Y-X)/2}\left[G_1(Y)\right]\,\Theta_{Y/2}\left[G_2(X-Y)\right]. \tag{1.2}$$

We turn now to Rieffel quantization [25, 26]. Let us denote by $\mathcal{A}^\infty$ the family of elements f such that the mapping $\Xi \ni X \mapsto \Theta_X(f) \in \mathcal{A}$ is C^∞. It is a dense *-algebra of $\mathcal{A}$ and also a Fréchet algebra with the family of semi-norms

$$|f|^k_{\mathcal{A}} := \sum_{|\alpha|\le k}\|\,\partial^\alpha_X\left[\Theta_X(f)\right]_{X=0}\|_{\mathcal{A}} \equiv \sum_{|\alpha|\le k}\|\,\delta^\alpha(f)\,\|_{\mathcal{A}}\,, \qquad k\in\mathbb{N}\,. \tag{1.3}$$

To quantize the above structure, one keeps the involution unchanged but introduce on $\mathcal{A}^\infty$ the product

$$f\#g := 2^{2n}\int_\Xi\int_\Xi dY\,dZ\,e^{2i[\![Y,Z]\!]}\,\Theta_Y(f)\,\Theta_Z(g)\,, \tag{1.4}$$

suitably defined by oscillatory integral techniques. One gets a *-algebra $(\mathcal{A}^\infty,\#,^*)$, which admits a C^*-completion $\mathfrak{A}$ in a C^*-norm $\|\cdot\|_{\mathfrak{A}}$ defined by Hilbert module techniques. The action Θ leaves $\mathcal{A}^\infty$ invariant and extends to a strongly continuous action of the C^*-algebra $\mathfrak{A}$, that will also be denoted by Θ. The space $\mathfrak{A}^\infty$ of C^∞-vectors coincide with $\mathcal{A}^\infty$, even topologically, i.e. the family (1.3) on $\mathcal{A}^\infty = \mathfrak{A}^\infty$ is equivalent to the family of semi-norms

$$|f|^k_{\mathfrak{A}} := \sum_{|\alpha|\le k}\|\,\partial^\alpha_X\left[\Theta_X(f)\right]_{X=0}\|_{\mathfrak{A}} \equiv \sum_{|\alpha|\le k}\|\,\delta^\alpha(f)\,\|_{\mathfrak{A}}\,, \qquad k\in\mathbb{N}\,. \tag{1.5}$$

An important particular case is obtained when $\mathcal{A}$ is a C^*-algebra of bounded uniformly continuous functions on the group Ξ, which is invariant under translations, i.e. if $\mathsf{h}\in\mathcal{A}$ and $X\in\Xi$, then $[\mathcal{T}_X(\mathsf{h})](\cdot) :=$

$\mathsf{h}(\cdot - X) \in \mathcal{A}$. In this case Rieffel's construction, done for $\Theta = \mathcal{T}$, reproduces essentially the standard Weyl calculus; we are going to use the special notation $\sharp$ (instead of $\#$) for the corresponding composition law.

Following [25], we introduce the Fréchet space $\mathcal{S}(\Xi;\mathfrak{A}^\infty)$ composed of smooth functions $F:\Xi\to\mathcal{A}^\infty=\mathfrak{A}^\infty$ with derivatives that decay rapidly with respect to all $|\cdot|^k_{\mathcal{A}}$. The relevant seminorms on $\mathcal{S}(\Xi;\mathfrak{A}^\infty)$ are now $\{\|\cdot\|^{k,\beta,N}_{\mathfrak{A}} \mid k,N\in\mathbb{N},\ \beta\in\mathbb{N}^{2n}\}$ where

$$\| F \|^{k,\beta,N}_{\mathfrak{A}} := \sup_{X\in\Xi}\{(1+|X|)^N |(\partial^\beta F)(X)|^k_{\mathfrak{A}}\}, \tag{1.6}$$

and the index $\mathfrak{A}$ can be replaced by $\mathcal{A}$. On it (and on many other larger spaces) one can define obvious actions $\mathfrak{T} := \mathcal{T}\otimes 1$ and $\mathcal{T}\otimes\Theta$ of the vector spaces Ξ and $\Xi\times\Xi$, respectively. Explicitly, for all $A,Y,X\in\Xi$, one sets $[\mathfrak{T}_A(F)](X) := F(X-A)$ and $[(\mathcal{T}_A\otimes\Theta_Y)F](X) := \Theta_Y[F(X-A)]$. Then on $\mathcal{S}(\Xi;\mathfrak{A}^\infty)$ one can introduce the composition law

$$(F_1\Box F_2)(X) = 2^{2n}\int_\Xi\int_\Xi dAdB\, e^{-2i[\![A,B]\!]}\,[\mathfrak{T}_A(F_1)](X)\#\,[\mathfrak{T}_B(F_2)](X) = \tag{1.7}$$

$$= 2^{4n}\int_\Xi\int_\Xi\int_\Xi\int_\Xi dY dZ\, dA\, dB\, e^{2i[\![Y,Z]\!]}\,e^{-2i[\![A,B]\!]}\,[(\mathcal{T}_A\otimes\Theta_Y)(F_1)](X)\,[(\mathcal{T}_B\otimes\Theta_Z)(F_2)](X)\,. \tag{1.8}$$

If the involution is given by $F^\Box(X) := F(X)^*,\ \forall X\in\Xi$, it can be shown that one gets a Fréchet *-algebra.

Remark 1.1. We recall that $\mathcal{A}^\infty = \mathfrak{A}^\infty$, even topologically, but the algebraic structures are different. When the forthcoming arguments will involve the non-commutative composition $\#$, in order to be more suggestive, we will use the notation $\mathcal{S}(\Xi;\mathfrak{A}^\infty)$. In other situations the notation $\mathcal{S}(\Xi;\mathcal{A}^\infty)$ will be more natural. For instance, it is easy to check that $\mathcal{S}(\Xi;\mathcal{A}^\infty)$ is a (dense) *-subalgebra of the Banach *-algebra $\left(L^1(\Xi;\mathcal{A}),\diamond,^\diamond,\|\cdot\|_1\right)$, which is defined in terms of the product $\cdot$ on $\mathcal{A}$ and has a priori nothing to do with the composition law $\#$.

Remark 1.2. One can also take into account $BC_{\rm u}(\Xi;\mathfrak{A})$, the C^*-algebra of all bounded and uniformly continuous functions $F:\Xi\to\mathfrak{A}$. Rieffel quantization can also be applied to the new classical data $(BC_{\rm u}(\Xi;\mathfrak{A}),\mathfrak{T},\Xi,-[\![\cdot,\cdot]\!])$, getting essentially (1.7) as the corresponding composition law (oscillatory integrals are needed). By using the second part (1.8) of the formula, this can also be regarded as the Rieffel composition constructed starting with the extended classical data $(BC_{\rm u}(\Xi;\mathcal{A}),\mathcal{T}\otimes\Theta,\Xi\times\Xi,\overline{\kappa}\otimes\kappa)$. All these will not be needed here. But we are going to use below the fact that for elements $f,g\in\mathfrak{A}^\infty$, $\mathsf{h},\mathsf{k}\in\mathcal{S}(\Xi)$ one has $(\mathsf{h}\otimes f)\Box(\mathsf{k}\otimes g) = (\mathsf{k}\sharp\,\mathsf{h})\otimes(f\#g)$, so $\Box$ can be seen as the tensor product between $\#$ and the law opposite to $\sharp$.

2 Modulation mappings and spaces

Definition 2.1. *On* $\mathcal{S}(\Xi;\mathfrak{A}^\infty)$ *we introduce* the global modulation mappings

$$[M(F)](X) := \int_\Xi dY\, e^{-i[\![X,Y]\!]}\,\Theta_Y[F(Y)] \tag{2.1}$$

and

$$\left[M^{-1}(G)\right](X) = \int_\Xi dY\, e^{-i[\![X,Y]\!]}\,\Theta_{-X}[G(Y)]\,. \tag{2.2}$$

To give a precise meaning to these relations, we introduce *the symplectic (partial) Fourier transform*

$$\mathfrak{F}\equiv\mathcal{F}\otimes 1:\mathcal{S}(\Xi;\mathfrak{A}^\infty)\to\mathcal{S}(\Xi;\mathcal{A}^\infty),\qquad (\mathfrak{F}F)(X) := \int_\Xi dY e^{-i[\![X,Y]\!]}F(Y)\,,$$

and force it to be L^2-unitary and satisfy $\mathfrak{F}^2 = \mathrm{id}$ by a suitable choice of Lebesgue measure dY on Ξ. Defining also C by $[C(F)](X) := \Theta_X[F(X)]$, we have $M = \mathfrak{F}\circ C$ and $M^{-1} = C^{-1}\circ\mathfrak{F}$.

Proposition 2.2. *The mapping*

$$M:\left(\mathcal{S}\left(\Xi;\mathfrak{A}^{\infty}\right),\Box,{}^{\Box}\right)\rightarrow\left(\mathcal{S}\left(\Xi;\mathcal{A}^{\infty}\right),\diamond,{}^{\diamond}\right)$$

is an isomorphism of Fréchet **-algebras and* M^{-1} *is its inverse.*

Proof. The partial Fourier transform is an isomorphism. One also checks that C is an isomorphism of $\mathcal{S}\left(\Xi;\mathfrak{A}^{\infty}\right)$; this follows from the explicit form of the seminorms on $\mathcal{S}(\Xi;\mathfrak{A}^{\infty})$, from the fact that Θ_X is isometric and from the formula

$$\partial^{\beta}\left[\Theta_X(F(X))\right]=\sum_{\gamma\le\beta}C_{\beta\gamma}\Theta_X\{\delta^{\gamma}\left[(\partial^{\beta-\gamma}F)(X)\right]\}.$$

With this remarks we conclude that $M=\mathfrak{F}\circ C$ and $M^{-1}=C^{-1}\circ\mathfrak{F}$ are reciprocal topological linear isomorphisms.

We still need to show that M is a *-morphism. *For the involution*:

$$[M(F)]^{\diamond}(X)=\left\{\int_{\Xi}dY\,e^{i[\![X,Y]\!]}\Theta_Y\left[F\left(Y\right)\right]\right\}^{*}=\int_{\Xi}dY\,e^{-i[\![X,Y]\!]}\Theta_Y\left[F\left(Y\right)^{*}\right]=\left[M\left(F^{\Box}\right)\right](X).$$

For the product: it is enough to show that $M^{-1}\left[M(F)\diamond M(G)\right]=F\Box G$ for all $F,G\in\mathcal{S}(\Xi;\mathfrak{A}^{\infty})$. One has (iterated integrals):

$$\left(M^{-1}\left[MF\diamond MG\right]\right)(X)=\int_{\Xi}dY_1\,e^{-i[\![X,Y_1]\!]}\,\Theta_{-X}\left\{\left[MF\diamond MG\right](Y_1)\right\}=$$

$$=\int_{\Xi}dY_1\,e^{-i[\![X,Y_1]\!]}\,\Theta_{-X}\left\{\int_{\Xi}dY_2\,e^{-i/2[\![Y_1,Y_2]\!]}\Theta_{(Y_2-Y_1)/2}\left[(MF)(Y_2)\right]\Theta_{Y_2/2}\left[(MG)(Y_1-Y_2)\right]\right\}=$$

$$=\int_{\Xi}dY_1\int_{\Xi}dY_2\,e^{-i[\![X,Y_1]\!]}\,e^{-i/2[\![Y_1,Y_2]\!]}\,\Theta_{-X}\left\{\Theta_{(Y_2-Y_1)/2}\left[(MF)(Y_2)\right]\Theta_{Y_2/2}\left[(MG)(Y_1-Y_2)\right]\right\}=$$

$$=\int_{\Xi}dY_1\int_{\Xi}dY_2\,e^{-i[\![X,Y_1]\!]}\,e^{-i/2[\![Y_1,Y_2]\!]}\,\cdot$$

$$\cdot\,\Theta_{(Y_2-Y_1)/2-X}\left\{\int_{\Xi}dY_3\,e^{-i[\![Y_2,Y_3]\!]}\,\Theta_{Y_3}\left[F\left(Y_3\right)\right]\right\}\Theta_{Y_2/2-X}\left\{\int_{\Xi}dY_4\,e^{-i[\![Y_1-Y_2,Y_4]\!]}\,\Theta_{Y_4}\left[G\left(Y_4\right)\right]\right\}=$$

$$=\int_{\Xi}dY_1\int_{\Xi}dY_2\int_{\Xi}dY_3\int_{\Xi}dY_4\,e^{-i[\![X,Y_1]\!]}\,e^{-i/2[\![Y_1,Y_2]\!]}\,e^{-i[\![Y_2,Y_3]\!]}e^{-i[\![Y_1-Y_2,Y_4]\!]}\,\cdot$$

$$\cdot\,\Theta_{Y_3+(Y_2-Y_1)/2-X}\left[F\left(Y_3\right)\right]\Theta_{Y_4+Y_2/2-X}\left[G\left(Y_4\right]\right)=$$

$$=2^{4n}\int_{\Xi}dY\int_{\Xi}dZ\int_{\Xi}dY_3\int_{\Xi}dY_4\,e^{-2i[\![X,Y_3-Y_4]\!]}\,e^{2i[\![Y,Z]\!]}\,e^{-2i[\![Y_3,Y_4]\!]}\,\Theta_Y\left[F\left(Y_3\right)\right]\Theta_Z\left[G\left(Y_4\right)\right].$$

For the last equality we made the substitution $Y=Y_3+\frac{1}{2}(Y_2-Y_1)-X,\ Z=Y_4+\frac{1}{2}Y_2-X$. Finally, setting $Y_3=X-A,\ Y_4=X-B$, we get

$$\left(M^{-1}\left[MF\diamond MG\right]\right)(X)=[F\Box G](X)=$$

$$=2^{4n}\int_{\Xi}dY\int_{\Xi}dZ\int_{\Xi}dA\int_{\Xi}dB\,e^{-2i[\![A,B]\!]}\,e^{2i[\![Y,Z]\!]}\,\Theta_Y\left[F\left(X-A\right)\right]\Theta_Z\left[G\left(X-B\right)\right].$$

$\Box$

Part of the nature of the modulation mapping is revealed by *localization*. This means to consider $M(F)$ for decomposable functions $F(\cdot) = (\mathsf{h} \otimes f)(\cdot) := \mathsf{h}(\cdot)f$, with $\mathsf{h} \in \mathcal{S}(\Xi)$ and $f \in \mathfrak{A}^\infty$ and then to freeze h (very often called *the window*), using this to examine f. For $\mathsf{h} \in \mathcal{S}(\Xi)$ we define $J_\mathsf{h} : \mathfrak{A}^\infty \to \mathcal{S}(\Xi; \mathfrak{A}^\infty)$ and $\widetilde{J}_\mathsf{h} : \mathcal{S}(\Xi; \mathfrak{A}^\infty) \to \mathfrak{A}^\infty$ by

$$J_\mathsf{h}(f) := \mathsf{h} \otimes f, \qquad \widetilde{J}_\mathsf{h}(F) := \int_\Xi dY\, \overline{\mathsf{h}(Y)} F(Y).$$

Definition 2.3. The localized modulation map defined by $\mathsf{h} \in \mathcal{S}(\Xi) \setminus \{0\}$ *is the linear injection*

$$M_\mathsf{h} : \mathfrak{A}^\infty \to \mathcal{S}(\Xi; \mathcal{A}^\infty), \qquad M_\mathsf{h}(f) := (M \circ J_\mathsf{h})(f) = M(\mathsf{h} \otimes f). \tag{2.3}$$

Explicitly, we get

$$[M_\mathsf{h}(f)](X) = \int_\Xi dY\, e^{-i[\![X,Y]\!]} \mathsf{h}(Y) \Theta_Y(f), \tag{2.4}$$

which can also be expressed in terms of symplectic Fourier transforms and convolution: $M_\mathsf{h}(f) = \mathfrak{F}[\mathsf{h}\Theta_f] = \widehat{\mathsf{h}} * \widehat{\Theta_f}$, where one uses a previous notation $\Theta_f : \Xi \to \mathcal{A},\ \Theta_f(X) := \Theta_X(f)$.

We also set $\widetilde{M}_\mathsf{h} = \widetilde{J}_\mathsf{h} \circ M^{-1}$. Obviously one has

$$\widetilde{M}_\mathsf{k} M_\mathsf{h} = \widetilde{J}_\mathsf{k} J_\mathsf{h} = \int_\Xi dX \overline{\mathsf{k}(X)} \mathsf{h}(X)\, \mathrm{id} =: \langle \mathsf{k}, \mathsf{h} \rangle_\Xi\, \mathrm{id}\,,$$

a particular case of which can be regarded as *an inversion formula*:

$$f = \frac{1}{\| \mathsf{h} \|_\Xi^2} \widetilde{M}_\mathsf{h} M_\mathsf{h} f\,. \tag{2.5}$$

Fortunately, the localized modulation maps can be extended to C^*-morphisms.

Corollary 2.4. *If* $\mathsf{h} \sharp \mathsf{h} = \mathsf{h} = \overline{\mathsf{h}} \in \mathcal{S}(\Xi) \setminus \{0\}$, *then* $M_\mathsf{h} : (\mathfrak{A}^\infty, \#, ^*) \to (\mathcal{S}(\Xi; \mathcal{A}^\infty), \diamond, ^\diamond)$ *is a* *-*monomorphism. It extends to a* C^*-*algebraic monomorphism* $M_\mathsf{h} : \mathfrak{A} \to \mathcal{A} \rtimes_\Theta^\kappa \Xi$.

Proof. The first assertion follows from Proposition 2.2 and from the fact that, if h is a non-null self-adjoint idempotent in $(\mathcal{S}(\Xi), \sharp, \bar{\cdot})$, then $J_\mathsf{h} : (\mathfrak{A}^\infty, \#, ^*) \to \left(\mathcal{S}(\Xi; \mathfrak{A}^\infty), \square, ^\square\right)$ is a *-monomorphism.

Taking into account the fact that $\mathcal{S}(\Xi; \mathcal{A}^\infty)$ is a *-subalgebra of $L^1(\Xi; \mathcal{A})$, which is in its turn a *-subalgebra of the C^*-algebra $\mathcal{A} \rtimes_\Theta^\kappa \Xi$, we examine the injective *-morphism $M_\mathsf{h} : \mathfrak{A}^\infty \to \mathcal{A} \rtimes_\Theta^\kappa \Xi$. We claim that it is isometric when on $\mathfrak{A}^\infty$ one considers the norm $\| \cdot \|_\mathfrak{A}$; this would insure that it can be extended to an injective *-morphism on $\mathfrak{A}$. By the paragraph 3.1.6 in [5], this follows if $\mathfrak{A}^\infty$ is invariant under the C^∞ functional calculus of $\mathfrak{A}$. But this property is obtained by straightforward extensions of the results of Subsection 3.2.2 in [6], writing $\mathfrak{A}^\infty$ as the intersection of domains of arbitrarily large products of the norm-closed derivations $\{\delta_j\}_{j=1,\dots,2n}$ associated to the $2n$-parameter group Θ of *-automorphisms of $\mathfrak{A}$. $\square$

We use the mappings M and M_h to transport structure. If $\| \cdot \| : \mathcal{S}(\Xi; \mathcal{A}^\infty) \to \mathbb{R}_+$ is a norm, we define a new one by

$$\| \cdot \|^M : \mathcal{S}(\Xi; \mathfrak{A}^\infty) \to \mathbb{R}_+, \qquad \| F \|^M := \| M(F) \|\,. \tag{2.6}$$

Assume now that a function $\mathsf{h} \in \mathcal{S}(\Xi) \setminus \{0\}$ (a window) is given. We define the norms

$$\| \cdot \|_\mathsf{h}^M : \mathfrak{A}^\infty \to \mathbb{R}_+, \qquad \| f \|_\mathsf{h}^M := \| M_\mathsf{h}(f) \| = \| M(\mathsf{h} \otimes f \| = \| J_\mathsf{h}(f) \|^M\,. \tag{2.7}$$

Definition 2.5. *If* $\mathfrak{L}$ *denotes the completion of* $(\mathcal{S}(\Xi; \mathcal{A}^\infty), \| \cdot \|)$, *let us define* $\mathfrak{L}^M$ *to be the completion of* $\left(\mathcal{S}(\Xi; \mathfrak{A}^\infty), \| \cdot \|^M\right)$ *and* $\mathfrak{L}_\mathsf{h}^M$ *the completion of* $\left(\mathfrak{A}^\infty, \| \cdot \|_\mathsf{h}^M\right)$.

We call $\left(\mathfrak{L}_\mathsf{h}^M, \| \cdot \|_\mathsf{h}^M\right)$ the modulation space associated to the pair $(\mathfrak{L}, \mathsf{h})$.

Clearly, the normed spaces $\left(\mathcal{S}(\Xi;\mathfrak{A}^\infty), \|\cdot\|^M\right)$ and $(\mathcal{S}(\Xi;\mathcal{A}^\infty), \|\cdot\|)$ are isomorphic, while J_{h} is an isometric embedding of $\left(\mathfrak{A}^\infty, \|\cdot\|^M_{\mathsf{h}}\right)$ in $\left(\mathcal{S}(\Xi;\mathfrak{A}^\infty), \|\cdot\|^M\right)$ and M_{h} an isometric embedding of $\left(\mathfrak{A}^\infty, \|\cdot\|^M_{\mathsf{h}}\right)$ in $(\mathcal{S}(\Xi;\mathcal{A}^\infty), \|\cdot\|)$. By extension one gets mappings also denoted by $M : \mathfrak{L}^M \to \mathfrak{L}$ (an isomorphism) and $M_{\mathsf{h}} : \mathfrak{L}^M_{\mathsf{h}} \to \mathfrak{L}$ (an isometric embedding). Very often a Banach space $(\mathfrak{L}, \|\cdot\|)$ containing densely $\mathcal{S}(\Xi,\mathcal{A}^\infty)$ is given and one applies the procedure above to *induce* a Banach space $\mathfrak{L}^M_{\mathsf{h}}$ containing $\mathfrak{A}^\infty$ densely. The denseness of $\mathcal{S}(\Xi,\mathcal{A}^\infty)$ could be avoided using extra techniques, but this will not be done here.

Concerning the compatibility of norms with *-algebra structures we can say basically that, using a self-adjoint idempotent window, one induces Banach *-algebras from Banach *-algebras and C^*-algebras from C^*-algebras:

Proposition 2.6. *Assume that* $\mathsf{h} \neq 0$ *is a self-adjoint projection in* $(\mathcal{S}(\Xi), \sharp)$*, i.e.* $\mathsf{h} \,\sharp\, \mathsf{h} = \mathsf{h} = \overline{\mathsf{h}}$.

1. *If the involution* $^\diamond$ *in* $(\mathcal{S}(\Xi,\mathcal{A}^\infty), \|\cdot\|)$ *is isometric, the involution* * *in* $\left(\mathfrak{A}^\infty, \|\cdot\|^M_{\mathsf{h}}\right)$ *is also isometric and it extends to an isometric involution on* $\mathfrak{L}^M_{\mathsf{h}}$.

2. *If* $\|\cdot\|$ *is sub-multiplicative with respect to* $\diamond$*, then* $\|\cdot\|^M_{\mathsf{h}}$ *is sub-multiplicative with respect to* $\#$. *The completion* $\mathfrak{L}^M_{\mathsf{h}}$ *becomes a Banach algebra sent isometrically by* M_{h} *into the Banach algebra* $\mathfrak{L}$.

3. *If* $\|\cdot\|$ *is a* C^**-norm, then* $\|\cdot\|^M_{\mathsf{h}}$ *is also a* C^**-norm and* $\mathfrak{L}^M_{\mathsf{h}}$ *a* C^**-algebra, which can be identified with a* C^**-subalgebra of* $\mathfrak{L}$.

Proof. Obvious from the fact that M_{h} is a *-monomorphism. □

We now worry about the h-dependence of the Banach space $\mathfrak{L}^M_{\mathsf{h}}$. We say that the norm $\|\cdot\|$ on $\mathcal{S}(\Xi;\mathcal{A}^\infty)$ is *admissible* (and call the completion $\mathfrak{L}$ *an admissible Banach space*) if for any $\mathsf{h}, \mathsf{k} \in \mathcal{S}(\Xi) \setminus \{0\}$ the operator $R_{\mathsf{k},\mathsf{h}} := M_{\mathsf{k}}\widetilde{M}_{\mathsf{h}} : (\mathcal{S}(\Xi;\mathcal{A}^\infty), \|\cdot\|) \to (\mathcal{S}(\Xi;\mathcal{A}^\infty), \|\cdot\|)$ is bounded.

Proposition 2.7. *If for a fixed couple* (h, k) *the operator* $R_{\mathsf{k},\mathsf{h}}$ *is bounded, we get a continuous dense embedding* $\mathfrak{L}^M_{\mathsf{h}} \hookrightarrow \mathfrak{L}^M_{\mathsf{k}}$. *So, if* $\mathfrak{L}$ *is admissible, all the Banach spaces* $\{\mathfrak{L}^M_{\mathsf{h}} \mid \mathsf{h} \in \mathcal{S}(\Xi) \setminus \{0\}\}$ *are isomorphic.*

Proof. It is enough to show that for some positive constant $C(\mathsf{h},\mathsf{k})$ one has $\| f \|^M_{\mathsf{k}} \leq C(\mathsf{h},\mathsf{k}) \| f \|^M_{\mathsf{h}}$ for all $f \in \mathcal{S}(\Xi;\mathcal{A}^\infty)$. This follows from the assumption and from (2.5):

$$\| f \|^M_{\mathsf{k}} = \| M_{\mathsf{k}} f \| \leq \frac{1}{\| \mathsf{h} \|^2_{\Xi}} \left\| M_{\mathsf{k}} \left(\widetilde{M}_{\mathsf{h}} M_{\mathsf{h}} f \right) \right\| \leq \frac{\left\| M_{\mathsf{k}} \widetilde{M}_{\mathsf{h}} \right\|}{\| \mathsf{h} \|^2_{\Xi}} \| M_{\mathsf{h}} f \| = \frac{\| R_{\mathsf{k},\mathsf{h}} \|}{\| \mathsf{h} \|^2_{\Xi}} \| f \|^M_{\mathsf{h}} .$$

□

Admissibility is quite a common phenomenon, due to the strong regularity assumptions imposed on the windows. On the other hand, one deduces from Corollary 2.4 that $[\,\mathcal{A} \rtimes^\kappa_\Theta \Xi\,]^M_{\mathsf{h}} = \mathfrak{A}$ for any idempotent window h; in this case the norm is really h-independent.

We are going to examine concrete modulation spaces in a future article, in connection with applications.

3 Morphisms and representations

We investigate now the interplay between localized modulation maps and Ξ-morphisms. Let $(\mathcal{A}_j, \Theta_j, \Xi, \kappa)$, $j = 1, 2$, be two classical data and let $\mathcal{R} : \mathcal{A}_1 \to \mathcal{A}_2$ a Ξ-morphism, i. e. a C^*-morphism intertwining the two actions Θ_1, Θ_2. Then $\mathcal{R}$ acts coherently on C^∞-vectors ($\mathcal{R}[\mathcal{A}^\infty_1] \subset \mathcal{A}^\infty_2$) and extends to a morphism $\mathfrak{R} : \mathfrak{A}_1 \to \mathfrak{A}_2$ that also intertwines the corresponding actions. On the other hand, another C^*-morphism $\mathcal{R}^\rtimes : \mathcal{A}_1 \rtimes^\kappa_{\Theta_1} \Xi \to \mathcal{A}_2 \rtimes^\kappa_{\Theta_2} \Xi$ is assigned canonically to $\mathcal{R}$, uniquely defined by

$$\left[\mathcal{R}^\rtimes(F)\right](X) := \mathcal{R}\left[F(X)\right], \qquad \forall F \in L^1(\Xi;\mathcal{A}_1) .$$

If $\mathsf{h} \in \mathcal{S}(\Xi) \setminus \{0\}$ is a self-adjoint idempotent window, one defines as above the localized modulation maps $M^j_{\mathsf{h}} : \mathfrak{A}_j \to \mathcal{A}_j \rtimes^{\kappa}_{\Theta_j} \Xi$, $j = 1, 2$ (each involving the respective action). Then a short computation on the dense subspace $\mathcal{A}^{\infty}_1 = \mathfrak{A}^{\infty}_1$ of $\mathfrak{A}_1$ shows that

$$\mathcal{R}^{\rtimes} \circ M^1_{\mathsf{h}} \;=\; M^2_{\mathsf{h}} \circ \mathfrak{R} \,. \tag{3.1}$$

This should be put in the perspective of quantization of ideal and quotients. Suppose that one is given a short exact sequence

$$0 \longrightarrow \mathcal{J} \xrightarrow{\mathcal{P}} \mathcal{A} \xrightarrow{\mathcal{R}} \mathcal{B} \longrightarrow 0$$

composed of morphisms which are equivariant (in an obvious sense) with respect to the actions $\Theta^{\mathcal{J}}$, $\Theta^{\mathcal{A}}$, $\Theta^{\mathcal{B}}$. Since both Rieffel quantization [25] and the twisted crossed product [24] are exact functors, one gets obvious short exact sequences which are joined vertically in commutative diagrams by the localized modulation mappings $M^{\mathcal{J}}_{\mathsf{h}}$, $M^{\mathcal{A}}_{\mathsf{h}}$ and $M^{\mathcal{B}}_{\mathsf{h}}$:

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{J} \arrow[r, "\mathfrak{P}"] \arrow[d, "M^{\mathcal{J}}_{\mathsf{h}}"'] & \mathfrak{A} \arrow[r, "\mathfrak{R}"] \arrow[d, "M^{\mathcal{A}}_{\mathsf{h}}"] & \mathfrak{B} \arrow[r] \arrow[d, "M^{\mathcal{B}}_{\mathsf{h}}"] & 0 \\
0 \arrow[r] & \mathcal{J} \rtimes^{\kappa}_{\Theta^{\mathcal{J}}} \Xi \arrow[r, "\mathcal{P}^{\rtimes}"] & \mathcal{A} \rtimes^{\kappa}_{\Theta^{\mathcal{A}}} \Xi \arrow[r, "\mathcal{R}^{\rtimes}"] & \mathcal{B} \rtimes^{\kappa}_{\Theta^{\mathcal{B}}} \Xi \arrow[r] & 0
\end{tikzcd}$$

The notations are self-explaining and the reader can check easily the details.

We turn now to representations, always supposed to be non-degenerate. The natural Hilbert space realization of a twisted C^*-dynamical system $(\mathcal{A}, \Theta, \Xi, \kappa)$ is achieved by *covariant representations* $(r, T, \mathcal{H})$, where r is a representation of $\mathcal{A}$ in the C^*-algebra $\mathbb{B}(\mathcal{H})$ of all bounded linear operators in $\mathcal{H}$, T is a strongly continuous unitary projective representation in $\mathcal{H}$:

$$T(X)T(Y) = \kappa(Y, X)T(X + Y), \qquad \forall X, Y \in \Xi$$

and for any $Y \in \Xi$ and $g \in \mathcal{A}$ one has

$$T(Y)r(g)T(-Y) = r\left[\Theta_Y(g)\right].$$

Hilbert-space representations (the most general, actually) $r \rtimes T : \mathcal{A} \rtimes^{\kappa}_{\Theta} \Xi \to \mathbb{B}(\mathcal{H})$ are associated to covariant representations $(r, T, \mathcal{H})$ of $(\mathcal{A}, \Theta, \Xi, \kappa)$ by

$$(r \rtimes T)(G) := \int_{\Xi} dX\, r\left\{\Theta_{X/2}\left[G(X)\right]\right\} T(X), \qquad G \in L^1(\Xi; \mathcal{A})\,. \tag{3.2}$$

The constructions of this Section allow us to use covariant representations of the initial data in the representation theory of the quantized C^*-algebra $\mathfrak{A}$. Let $(r, T, \mathcal{H})$ be a covariant representations for $(\mathcal{A}, \Theta, \Xi, \kappa)$. and $\mathsf{h} \sharp \mathsf{h} = \overline{\mathsf{h}} = \mathsf{h} \in \mathcal{S}(\Xi) \setminus \{0\}$ any idempotent window. Composing $r \rtimes T : \mathcal{A} \rtimes^{\kappa}_{\Theta} \Xi \to \mathbb{B}(\mathcal{H})$ with the morphism $M_{\mathsf{h}} : \mathfrak{A} \to \mathcal{A} \rtimes^{\kappa}_{\Theta} \Xi$ (cf. Corollary 2.4) one gets the representation

$$(r \rtimes T)^M_{\mathsf{h}} := (r \rtimes T) \circ M_{\mathsf{h}} : \mathfrak{A} \to \mathbb{B}(\mathcal{H})\,, \tag{3.3}$$

which is given on $\mathfrak{A}^{\infty}$ by

$$(r \rtimes T)^M_{\mathsf{h}}(f) = \int_{\Xi} dX \int_{\Xi} dY\, e^{-i[\![X,Y]\!]}\, \mathsf{h}(Y)\, r\left\{\Theta_{Y+X/2}(f)\right\} T(X)\,. \tag{3.4}$$

If $r \rtimes T$ is faithful, $(r \rtimes T)^M_{\mathsf{h}}$ is faithful too, since M_{h} is injective. Unitary equivalence is preserved under the correspondence $(r, T) \mapsto (r \rtimes T)^M_{\mathsf{h}}$.

Remark 3.1. A simple computation shows that

$$T(Z)\,(r \rtimes T)^M_{\mathsf{h}}(f)\,T(-Z) = (r \rtimes T)^M_{\mathcal{T}_Z\mathsf{h}}(f), \qquad \forall Z \in \Xi,\ \forall f \in \mathfrak{A}^\infty, \tag{3.5}$$

and by density this also holds for $f \in \mathfrak{A}$. We recall the notation $(\mathcal{T}_Z\mathsf{h})(\cdot) = \mathsf{h}(\cdot - Z)$ and notice that h and $\mathcal{T}_Z\mathsf{h}$ are simultaneously self-adjoint projections. Thus, along the $\mathcal{T}$-orbits, the representations are unitarily equivalent.

Actually, starting with an arbitrary representation $\rho : \mathcal{A} \to \mathbb{B}(\mathcal{K})$, one can induce canonically a covariant representation $(r_\rho, T, \mathcal{H})$ of $(\mathcal{A}, \Theta, \Xi, \kappa)$, setting $\mathcal{H} := L^2(\Xi; \mathcal{K})$,

$$[r_\rho(f)\Phi]\,(X) := \rho\,[\Theta_X(f)]\,[\Phi(X)]\,, \qquad f \in \mathcal{A},\ X \in \Xi,\ \Phi \in L^2(\Xi; \mathcal{K})$$

and

$$[T(Y)\Phi]\,(X) := \kappa(Y, X)\Phi(X + Y), \qquad X, Y \in \Xi,\ \Phi \in L^2(\Xi; \mathcal{K})\,.$$

Then one associates the representations $\rho_{(M,\mathsf{h})} := (r_\rho \rtimes T)^M_{\mathsf{h}}$ of $\mathfrak{A}$ in $L^2(\Xi; \mathcal{K})$ indexed by the non-null self-adjoint projections in $(\mathcal{S}(\Xi), \sharp)$. It is straightforward to check that the correspondence $\rho \mapsto \rho_{(M,\mathsf{h})}$ preserves unitary equivalence.

Proposition 3.2. *1. For any idempotent window* $\mathsf{h}\,\sharp\,\mathsf{h} = \overline{\mathsf{h}} = \mathsf{h} \in \mathcal{S}(\Xi) \setminus \{0\}$ *and for each* $f \in \mathfrak{A}$ *one has*

$$\|f\|_{\mathfrak{A}} \,=\, \sup\left\{\|(r \rtimes T)\,[M_{\mathsf{h}}(f)]\,\|_{\mathbb{B}(\mathcal{H})} \mid (r, T, \mathcal{H}) \text{ covariant representation of } (\mathcal{A}, \Theta, \Xi, \kappa)\right\}. \tag{3.6}$$

2. Moreover, for any faithful representation $\rho : \mathcal{A} \to \mathbb{B}(\mathcal{K})$ *one has*

$$\|f\|_{\mathfrak{A}} \,=\, \|\rho_{(M,\mathsf{h})}(f)\|_{\mathbb{B}[L^2(\Xi;\mathcal{K})]}\,. \tag{3.7}$$

Proof. The two formulas follow from the fact that $M_{\mathsf{h}} : \mathfrak{A} \to \mathcal{A} \rtimes^\kappa_\Theta \Xi$ is an isometry and from the well-known forms of the universal and the reduced norm in twisted crossed products, that coincide since the group Ξ is Abelian [23]. □

Remark 3.3. Let us make some comments about how one could modify the definitions above. We are going to need the notation $[C_\alpha(F)]\,(X) := \Theta_{\alpha X}[F(X)]$, where $X \in \Xi$, $F \in \mathcal{S}(\Xi; \mathcal{A}^\infty)$ (or $F \in L^1(\Xi; \mathcal{A})$) and α is a real number. All these operations are isomorphisms and our previous C coincides with C_1. The traditional composition law in the twisted crossed product is not (1.2), but

$$(G_1 \diamond' G_2)(X) := \int_\Xi dY\,\kappa(X, Y)\,G_1(Y)\,\Theta_Y\,[G_2(X - Y)]\,.$$

Consequently, (3.2) should be replaced by $(r \rtimes' T)(G) := \int_\Xi dX\,r\,[G(X)]\;T(X)\,.$ The distinction is mainly an ordering matter and it corresponds to the distinction between the Weyl and the Kohn-Nierenberg forms of pseudodifferential theory. Applying $C_{1/2}$ leads to an isomorphism between the two algebraic structures. So, if we want to land in this second realization, we should replace $M = \mathfrak{F}\,C_1$ with $M' := C_{1/2}\,\mathfrak{F}\,C_1$, leading explicitly to $[M'(F)](X) := \int_\Xi dY\,e^{-i[\![X,Y]\!]}\,\Theta_{Y+X/2}\,[F\,(Y)]\,.$ Considering also a window $\mathsf{h} \in \mathcal{S}(\Xi)$ and setting $\mathsf{e}_X(\cdot) := e^{-i[\![X,\cdot]\!]}$, one gets the corresponding localized form

$$[M'_{\mathsf{h}}(f)](X) := \int_\Xi dY\,e^{-i[\![X,Y]\!]}\,\mathsf{h}(Y)\Theta_{X/2}\,[\Theta_Y(f)] = \langle\,\overline{\mathsf{h}}\,, \mathsf{e}_X\Theta_{X/2}\,[\Theta_\cdot(f)]\,\rangle_\Xi\,.$$

These new modulation mappings are closer to those used in studying the standard Weyl calculus; they can also be used in our framework to induce modulation-type spaces.

4 The abelian case

If $\mathcal{A}$ is Abelian, by Gelfand theory, it is isomorphic (and will be identified) to $\mathcal{C}(\Sigma)$, the C^*-algebra of all complex continuous functions on the locally compact space Σ which converge to zero at infinity. The space Σ is a homeomorphic copy of the Gelfand spectrum of $\mathcal{A}$ and it is compact iff $\mathcal{A}$ is unital. Then the group Θ of automorphisms is induced by an action (also called Θ) of Ξ by homeomorphisms of Σ. We are going to use the convention

$$[\Theta_X(f)](\sigma) := f[\Theta_X(\sigma)], \qquad \forall\, \sigma \in \Sigma,\ X \in \Xi,\ f \in \mathcal{A},$$

as well as the notation $\Theta_X(\sigma) = \Theta(X,\sigma) = \Theta_\sigma(X)$ for the X-transform of the point σ.

We discuss shortly "orthogonality matters". On Σ we also consider a Θ-invariant measure $d\sigma$ and work with scalar products of the form

$$\langle f,g\rangle_\Sigma := \int_\Sigma d\sigma \overline{f(\sigma)} g(\sigma), \qquad \langle F,G\rangle_{\Xi\times\Sigma} := \int_\Xi \int_\Sigma dX d\sigma \overline{F(X,\sigma)} G(X,\sigma).$$

The relationship between the spaces $\mathcal{S}(\Xi;\mathcal{A}^\infty)$ and $L^2(\Xi\times\Sigma)$ depends on the assumptions we impose on $(\Sigma, d\sigma)$. If $d\sigma$ is a finite measure, for instance, one has $\mathcal{S}(\Xi;\mathcal{A}^\infty) \subset L^2(\Xi\times\Sigma)$. Anyhow, the modulation map can be defined independently on $L^2(\Xi\times\Sigma)$. We record this, but it will not be used and the existence of $d\sigma$ will not be needed below.

Proposition 4.1. *One has* the orthogonality relations *valid for* $F,G \in L^2(\Sigma\times\Xi)$*:*

$$\langle M(F), M(G)\rangle_{\Xi\times\Sigma} = \langle F,G\rangle_{\Xi\times\Sigma}. \tag{4.1}$$

Thus the operator $M: L^2(\Xi\times\Sigma) \to L^2(\Xi\times\Sigma)$ *is unitary.*

Proof. It is enough to note that $M = \mathfrak{F}\circ C$ and to use the fact that $\mathfrak{F}$ and C are isomorphisms of $L^2(\Sigma\times\Xi)$ if $d\sigma$ is Θ-invariant. $\square$

Assuming that $\mathcal{A} \equiv \mathcal{C}(\Sigma)$ is Abelian, we set $\mathfrak{A} =: \mathfrak{C}(\Sigma)$ for the (non-commutative) Rieffel C^*-algebra associated to $\mathcal{C}(\Sigma)$ by quantization and $\mathcal{C}^\infty(\Sigma) = \mathfrak{C}^\infty(\Sigma)$ for the (common) space of smooth vectors under the action Θ. For each $\sigma\in\Sigma$, we introduce a concrete covariant representation $\big(r_\sigma, T, L^2(\Xi)\big)$ of the twisted dynamical system $(\mathcal{C}(\Sigma),\Theta,\Xi,\kappa)$ by

$$r_\sigma : \mathcal{C}(\Sigma) \to \mathbb{B}\left[L^2(\Xi)\right], \qquad [r_\sigma(g)\Phi](X) := g[\Theta_X(\sigma)]\,\Phi(X) \tag{4.2}$$

and

$$T(Y): L^2(\Xi) \to L^2(\Xi), \qquad [T(Y)\Phi](X) := \kappa(Y,X)\Phi(X+Y). \tag{4.3}$$

It is induced from the one dimensional representation $\rho_\sigma : \mathcal{C}(\Sigma) \to \mathbb{B}(\mathbb{C}) \cong \mathbb{C},\ \rho_\sigma(f) := f(\sigma)$. The general procedure of the preceding section provides a family of representations $(r_\sigma \rtimes T)^M_{\mathsf{h}}$ of $\mathfrak{A}$ in the Hilbert space $L^2(\Xi)$, indexed by the non-null projections of $(\mathcal{S}(\Xi),\sharp)$.

To connect these representations with rather familiar Weyl-type operators, we need first to recall somehow informally some basic facts about the standard Weyl quantization $\mathsf{f} \mapsto \mathfrak{Op}(\mathsf{f})$ [10]. We assume that $\Xi = \mathscr{X}\times\mathscr{X}^*$. The action of $\mathfrak{Op}(f)$ on $\mathcal{S}(\mathscr{X})$ or $\mathcal{H} := L^2(\mathscr{X})$ (under various assumptions and with various interpretations) is given by

$$[\mathfrak{Op}(\mathsf{f})v](x) := \int_{\mathscr{X}}\int_{\mathscr{X}^*} dy\, d\xi\, e^{i(x-y)\cdot\xi}\, \mathsf{f}\left(\frac{x+y}{2},\xi\right) v(y). \tag{4.4}$$

We recall that $\mathfrak{Op}(\mathsf{f}\,\sharp\,\mathsf{g}) = \mathfrak{Op}(\mathsf{f})\mathfrak{Op}(\mathsf{g})$ and $\mathfrak{Op}(\mathsf{f}^*) = \mathfrak{Op}(\mathsf{f})^*$. It is useful to introduce the family of unitary operators

$$\mathfrak{op}(X) = \mathfrak{Op}(\mathsf{e}_X), \qquad \mathsf{e}_X(Y) := e^{-i[\![X,Y]\!]}, \quad \forall X,Y\in\Xi. \tag{4.5}$$

satisfying $\mathfrak{op}(X)\,\mathfrak{op}(Y) = \kappa(X,Y)\,\mathfrak{op}(X+Y),\ \forall X,Y\in\Xi$.

Using these, one can introduce a family $\{\mathfrak{Op}_\sigma \mid \sigma\in\Sigma\}$ of Schrödinger-type representations of $\mathfrak{C}(\Sigma)$ in the Hilbert space $\mathcal{H}=L^2(\mathscr{X})$, indexed by the points of the dynamical system. They are given for $f\in\mathfrak{C}^\infty(\Sigma)$ by $\mathfrak{Op}_\sigma(f):=\mathfrak{Op}[f\circ\Theta_\sigma]$; using oscillatory integrals one may write

$$[\mathfrak{Op}_\sigma(f)u](x)=\int_{\mathscr{X}}\int_{\mathscr{X}^*}dyd\xi\, e^{i(x-y)\xi}\, f\left[\Theta_{\left(\frac{x+y}{2},\xi\right)}(\sigma)\right]u(y),\qquad u\in L^2(\mathscr{X})\,. \tag{4.6}$$

The extension from $\mathfrak{C}^\infty(\Sigma)$ to $\mathfrak{C}(\Sigma)$ is slightly non-trivial, but it is explained in [19]. Note that if σ and σ' belong to the same Θ-orbit, the representations $\mathfrak{Op}_\sigma$ and $\mathfrak{Op}_{\sigma'}$ are unitarily equivalent. $\mathfrak{Op}_\sigma$ is faithful if and only if the orbit generated by σ is dense. The justifications and extra details can be found in [19].

Remark 4.2. It is easy to see that $\left(\mathfrak{Op}_\sigma,\mathfrak{op},L^2(\mathscr{X})\right)$ is a covariant representation of $(\mathfrak{C}(\Sigma),\Theta,\Xi,\kappa)$. This follows applying $\mathfrak{Op}$ to the relations

$$\mathsf{e}_Y\sharp\,\mathsf{e}_Z=\kappa(Y,Z)\,\mathsf{e}_{Y+Z},\qquad \mathsf{e}_Y\sharp\,(f\circ\Theta_\sigma)\sharp\,\mathsf{e}_{-Y}=[\Theta_Y(f)]\circ\Theta_\sigma\,.$$

We would like now to make the connection between $\mathfrak{Op}_\sigma$ and $(r_\sigma\rtimes T)^M_{\mathsf{h}}$ for convenient idempotent windows. This needs some preparations involving the Bargmann transform.

For various types of vectors $u,v:\mathscr{X}\to\mathbb{C}$ we define *the Wigner transform* (V) and *the Fourier-Wigner transform* (W) by $W_{u,v}(X)=\langle u,\mathfrak{op}(X)v\rangle_{\mathscr{X}}$ and $V_{u,v}=\mathcal{F}W_{u,v}$. Their important role is shown by the relations

$$\langle u,\mathfrak{Op}(\mathsf{f})v\rangle_{\mathscr{X}}=\int_\Xi dX\,\mathsf{f}(X)\,V_{u,v}(X),\qquad \langle u,\mathfrak{Op}(\mathsf{f})v\rangle_{\mathscr{X}}=\int_\Xi dX\,(\mathcal{F}\mathsf{f})\,(X)\,W_{u,v}(X)\,. \tag{4.7}$$

Let us fix $v\in\mathcal{S}(\mathscr{X})$ with $\parallel v\parallel=1$. For any $Y\in\Xi$ we define $v(Y):=\mathfrak{op}(-Y)v\in\mathcal{H}$ (*the family of coherent vectors associated to* v). The isometric mapping $\mathcal{U}_v:L^2(\mathscr{X})\to L^2(\Xi)$ given by

$$(\mathcal{U}_v u)\,(X):=\langle v(X),u\rangle_{\mathscr{X}}=\langle v,\mathfrak{op}(X)u\rangle_{\mathscr{X}}=W_{v,u}(X) \tag{4.8}$$

is called *the (generalized) Bargmann transformation* corresponding to the family of coherent states $\{v(X)\mid X\in\Xi\}$. Its adjoint is given by

$$\mathcal{U}_v^*\Phi=\int_\Xi dY\,\Phi(Y)v(Y),\qquad \forall\Phi\in L^2(\Xi)\,. \tag{4.9}$$

We also set $\mathbb{U}_v[T]:=\mathcal{U}_vT\mathcal{U}_v^*$ for any $T\in\mathbb{B}\left[L^2(\mathscr{X})\right]$. Now take $\mathsf{h}\equiv\mathsf{h}(v):=V_{v,v}$ with explicit form

$$[\mathsf{h}(v)](x,\xi)=\int_{\mathscr{X}}dy\,e^{iy\cdot\xi}\,\overline{u\left(x+\frac{y}{2}\right)}u\left(x-\frac{y}{2}\right)\,.$$

Then $\mathfrak{Op}(\mathsf{h}(v))$ will be the rank-one projection $|v\rangle\langle v|$, thus $\mathsf{h}(v)\sharp\mathsf{h}(v)=\overline{\mathsf{h}(v)}=\mathsf{h}(v)\,.$

Proposition 4.3. *One has on* $\mathfrak{C}(\Sigma)$

$$(r_\sigma\rtimes T)^M_{\mathsf{h}(v)}=\mathbb{U}_v\circ\mathfrak{Op}_\sigma. \tag{4.10}$$

Proof. By density, it is enough to compute on $\mathcal{C}^\infty(\Sigma)$. Using successively the expressions of $\mathcal{U}_v,\mathfrak{Op}_\sigma,\mathcal{U}_v^*$, formulas (4.5) and (4.7) and the fact that $\mathsf{h}(v)$ is real, we get

$$[\mathcal{U}_v\,\mathfrak{Op}_\sigma(f)\,\mathcal{U}_v^*\Phi]\,(X)=\langle v(X),\mathfrak{Op}_\sigma(f)\,\mathcal{U}_v^*\Phi\rangle_{\mathscr{X}}=\left\langle v(X),\mathfrak{Op}\,(f\circ\Theta_\sigma)\int_\Xi dY\,\Phi(Y)v(Y)\right\rangle_{\mathscr{X}}=$$

$$=\int_\Xi dY\,\Phi(Y)\langle v(X),\mathfrak{Op}\,(f\circ\Theta_\sigma)\,v(Y)\rangle_{\mathscr{X}}=\int_\Xi dY\,\Phi(Y)\langle v,\mathfrak{Op}(\mathsf{e}_X\sharp\,[f\circ\Theta_\sigma]\,\sharp\,\mathsf{e}_{-Y})v\rangle_{\mathscr{X}}=$$

$$=\int_\Xi dY\,\langle\mathsf{h}(v),\mathsf{e}_X\sharp\,[f\circ\Theta_\sigma]\,\sharp\,\mathsf{e}_{-Y}\rangle_\Xi\,\Phi(Y).$$

On the other hand, by (3.2), (4.2), (4.3), a change of variables and the form of $M_{\mathsf{h}(v)}$

$$\left[(r_\sigma \rtimes T)^M_{\mathsf{h}(v)}(f)(\Phi)\right](X) = \int_\Xi dZ\, \kappa(Z,X)\left[M_{\mathsf{h}(v)}(f)\right]\left(\Theta_{X+Z/2}(\sigma), Z\right)\Phi(X+Z) =$$

$$= \int_\Xi dY\, \kappa(Y,X)\left[M_{\mathsf{h}(v)}(f)\right]\left(\Theta_{(X+Y)/2}(\sigma), Y-X\right)\Phi(Y) =$$

$$= \int_\Xi dY\, \kappa(Y,X)\int_\Xi dZ\, e^{-i[\![Y-X,Z]\!]}\,[\mathsf{h}(v)](Z)\, f\left[\Theta_{Z+(X+Y)/2}(\sigma)\right]\Phi(Y)\,.$$

Thus it is enough to show that for all $f \in \mathcal{C}^\infty(\Sigma)$, $\mathsf{h} = \overline{\mathsf{h}} \in \mathcal{S}(\Xi)$, $X, Y \in \Xi$, $\sigma \in \Sigma$ one has

$$\langle \mathsf{h}, \mathsf{e}_X \sharp\, [f \circ \Theta_\sigma] \,\sharp\, \mathsf{e}_{-Y}\rangle_\Xi = \kappa(Y,X)\int_\Xi dZ\, e^{-i[\![Y-X,Z]\!]}\,\mathsf{h}(Z)\, f\left[\Theta_{Z+(X+Y)/2}(\sigma)\right]\,.$$

This amounts to

$$\left(\mathsf{e}_X \sharp\, [f \circ \Theta_\sigma] \,\sharp\, \mathsf{e}_{-Y}\right)(Z) = \kappa(Y,X)\, e^{-i[\![Y-X,Z]\!]}\, f\left[\Theta_\sigma\left(Z + (X+Y)/2\right)\right], \qquad \forall Z \in \Xi\,,$$

which follows from a straightforward computation of the left-hand side. □

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